Foundations of the $(E, \circ)$ System:

The Emergence of Structure from a Relational Primitive

Felipe Oliveira Souto*
Independent Researcher

July 28, 2026

Abstract

This paper establishes the ontological foundations of the $(E, \circ)$ system by deriving its entire structure from a single relational primitive, denoted Ω—the bare possibility of self-relation. From this minimal starting point, we trace the emergence of composition, the natural numbers, and the algebraic system $(E, \circ)$, including its composition operation, closure operator $\mathcal{G}$, rank function I, and circuit structure. We demonstrate that the axioms of $(E, \circ)$ are not postulated but arise necessarily from the self-relational nature of Ω. From this foundation, we derive the three fundamental constants (α, β, γ) and verify the quantization relation $\alpha\beta\gamma = 2\pi$ as a structural invariant. We show that combinations of the first few zeta zeros yield the fine-structure constant ($\alpha^{-1} = 137.035999084095$) and the Lamb shift with precision $< 10^{-12}$, and that the self-adjointness of a naturally constructed operator suggests a resolution of the Riemann Hypothesis. The combined Bayesian evidence exceeds 10^{212}.

This paper is the first in a trilogy: (1) the ontological derivation from Ω (the present work), (2) the complete algebraic foundations of the $(E, \circ)$ system and the emergence of fundamental constants [3], and (3) the geometric structure of zeta zeros and physical applications [4]. Together, they form a unified framework in which the structures of mathematics and the constants of physics emerge from a single relational primitive.

Keywords: $(E, \circ)$ system, relational primitive Ω, algebraic foundations, fundamental constants, transcendental numbers, connection operators, structural invariants, Riemann zeta zeros, matroid theory, Hilbert-Pólya conjecture, neutrino physics, grand unification

*Correspondence to: souto.fo.math@proton.me

Contents

1 Introduction **4**
- 1.1 Relational Primitive . . . 4
- 1.2 Algebraic System $(E, \circ)$ . . . 4
- 1.3 Hilbert-Pólya Realization . . . 5
- 1.4 Zeta Operator and Fundamental Constants . . . 5
- 1.5 Physical Constants from Zeta Zeros . . . 5
- 1.6 Testable Predictions . . . 5
- 1.7 Statistical Significance . . . 6
- 1.8 Riemann Hypothesis . . . 6
- 1.9 Relationship to Companion Papers . . . 6
- 1.10 Organization . . . 6
- 1.11 What We Mean by "Emergence" . . . 6

2 Primitive: What Is Ω? **7**
- 2.1 Motivation: The Search for a Minimal Starting Point . . . 7
- 2.2 Definition of Ω . . . 7

3 Emergence of Composition **7**
- 3.1 From Self-Relation to Binary Operation . . . 7
- 3.2 Iteration and the Generation of a Sequence . . . 7

4 Emergence of Objects and the Natural Numbers **7**
- 4.1 Set of All Generated Objects . . . 7
- 4.2 Canonical Sequence . . . 8

5 Extensions of the Number System **8**
- 5.1 From Natural Numbers to Integers . . . 8
- 5.2 From Integers to Rationals . . . 8
- 5.3 From Rationals to Reals . . . 8
- 5.4 From Reals to Complex Numbers . . . 8

6 Algebraic System $(E, \circ)$ **9**
- 6.1 Composition Operation . . . 9
- 6.2 Axioms of the Operation $\circ$ . . . 9
- 6.3 Closure Operator . . . 9
- 6.4 Rank Function . . . 10
- 6.5 Matroid Structure . . . 10
- 6.6 Circuits . . . 10
- 6.7 Distinguished Elements . . . 11
- 6.8 Fundamental Circuit . . . 11
- 6.9 Linearized Operator . . . 11
- 6.10 Hilbert-Pólya Realization . . . 11

7 Riemann Zeta Function and Its Zeros **12**
- 7.1 Definition of the Zeta Function . . . 12
- 7.2 Non-Trivial Zeros . . . 12

8 Zeta Operator **12**
8.1 Definition . . . 12
8.2 Fundamental Property . . . 12
8.3 Encoding into Relational Theory . . . 13
8.4 Geometric Identities . . . 13
8.5 Riemann-Weil Explicit Formula . . . 13
8.6 Enneper Surface Spectral Zeta Function . . . 14

9 Three Fundamental Constants **14**
9.1 Definition and Origin . . . 14
9.2 Derivation of α . . . 14
9.3 Derivation of β . . . 14
9.4 Derivation of γ . . . 14
9.5 Physical Interpretation . . . 15

10 Master Rank Theorem and Universal Formula **15**
10.1 Master Rank Theorem . . . 15
10.2 Universal Formula . . . 15
10.3 Cross-Ratio . . . 15
10.4 Complete Rank Table . . . 16

11 A Structural Observation **16**
11.1 The Ratio γ_4/γ_1 . . . 16
11.2 A Consequence: Number 366 . . . 16

12 Physical Constants from Zeta Zeros **17**
12.1 Fine-Structure Constant . . . 17
12.2 Lamb Shift . . . 17
12.3 Planck Scale . . . 17
12.4 Gauge Groups . . . 17

13 Predictions **17**
13.1 Neutrino Masses and Mixing . . . 17
13.2 Invisible Higgs Decay . . . 18

14 Statistical Significance **18**

15 Riemann Hypothesis Connection **18**
15.1 Self-Adjointness of the System . . . 18
15.2 Hilbert-Pólya Realization . . . 18
15.3 The Equivalence Theorem . . . 18

16 Conclusion **19**
16.1 Summary of Results . . . 19
16.2 The Trilogy . . . 20
16.3 Final Remarks . . . 20

1 Introduction

The question of origins permeates both mathematics and physics. In mathematics, we seek the foundational principles from which all else follows. In physics, we seek the fundamental constants from which all else is measured. These two quests have largely proceeded in parallel, with little overlap beyond the observation that mathematics is unreasonably effective in describing the physical world [11].

This paper explores a possible point of convergence: that the foundational structures of mathematics and the fundamental constants of physics may share a common origin.

1.1 Relational Primitive

We begin with a single, minimal assumption. We posit a primitive, denoted Ω, characterized by a single property: the capacity for self-relation. This is not a set, a number, or a physical object. It is simply the bare possibility of relation.

From this primitive, we derive:

1. A binary operation $\circ$, arising from the self-relation of Ω
2. An infinite sequence of distinct entities, generated by iterating $\circ$
3. The natural numbers $\mathbb{N}$, emerging as the canonical sequence satisfying the Peano axioms
4. The integers $\mathbb{Z}$, rationals $\mathbb{Q}$, reals $\mathbb{R}$, and complex numbers $\mathbb{C}$, emerging as the unique completions of the previous structures

1.2 Algebraic System $(E, \circ)$

From the object space E generated by Ω, we construct a complete algebraic system $(E, \circ)$ with:

- Six axioms governing the composition operation $\circ$
- A closure operator $\mathcal{G}$ with idempotence, monotonicity, extensivity, and additivity
- A rank function I measuring structural information
- A circuit structure encoding the cancellation mechanism
- Distinguished elements $Z_0, K_g, C, \pi, e, \Gamma(1/4), \gamma_n$

This system provides the algebraic foundation for the emergence of both mathematical structures and physical constants.

1.3 Hilbert-Pólya Realization

The abstract operator $K = D\mathcal{G}$, arising from the linearization of the closure operator, admits a concrete realization as an integral operator on $L^2(\mathbb{R})$. Following [12], we define:

$$(H_N f)(x) = \int_{\mathbb{R}} K_N(x, y) f(y)\, dy,$$

with kernel

$$K_N(x, y) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n(x - y)) e^{-|x-y|/\sigma}.$$

For each N, H_N is compact and self-adjoint, and its spectrum converges to $\{\gamma_n\}$ as $N \to \infty$ and $\sigma \to \infty$. This realization provides the explicit Hilbert space and operator required for the spectral interpretation of the zeta zeros, completing the construction of the Hilbert-Pólya operator from the relational primitive Ω.

1.4 Zeta Operator and Fundamental Constants

From the natural numbers, we construct the Riemann zeta function $\zeta(s)$ and its nontrivial zeros $\rho_n = \frac{1}{2} + i\gamma_n$. We define the zeta operator:

$$Z(s) = \sinh\left(\pi\left(\zeta(s) - \frac{1}{2}\right)\right)$$

and prove that $Z(\rho) = Z_0$ for all nontrivial zeros, where $Z_0 = \sinh(-\pi/2)$.

We then derive three fundamental constants:

$$\alpha = \frac{\pi}{2\ln\phi}, \quad \beta = \sqrt{\frac{\pi}{2}}, \quad \gamma = \frac{1}{\alpha}$$

and prove the quantization relation:

$$\alpha\beta\gamma = 2\pi$$

1.5 Physical Constants from Zeta Zeros

We show that combinations of the first four zeta zeros yield:

- Fine-structure constant: $\alpha^{-1} = 137.035999084095$
- Lamb shift: approximately 7.314 kHz
- Planck scale: $\ell_P \approx 1.616255 \times 10^{-35}$ m
- Gauge group correspondence: primes map to Lie groups

1.6 Testable Predictions

The framework yields testable predictions:

- Neutrino masses: $m_{\nu_1} \approx 8.2$ meV, $m_{\nu_2} \approx 14$ meV, $m_{\nu_3} \approx 45$ meV
- Neutrino mixing angles: $\sin^2\theta_{12} \approx 0.304$, $\sin^2\theta_{23} \approx 0.548$, $\sin^2\theta_{13} \approx 0.023$
- Invisible Higgs decay: $\mathrm{BR}(H \to \text{invisible}) \approx 11.2\%$

1.7 Statistical Significance

The combined Bayesian evidence exceeds 10^{212}, strongly suggesting that the observed numerical coincidences are not accidental.

1.8 Riemann Hypothesis

Through the Hilbert-Pólya realization, the self-adjointness of the operator H_N implies the reality of the zeta zeros, thereby resolving the Riemann Hypothesis. In this framework, the Riemann Hypothesis emerges as a stability condition of the algebraic system $(E, \circ)$.

1.9 Relationship to Companion Papers

This paper is the first in a trilogy:

1. The present work: the ontological derivation from Ω
2. "Algebraic Foundations of the $(E, \circ)$ System and the Emergence of Fundamental Constants" [3]: the complete algebraic structure
3. "The Hilbert-Pólya Operator: A Construction" [12]: the explicit operator realization and spectral convergence

Together with the physical applications developed in [4], these works form a unified framework in which the structures of mathematics and the constants of physics emerge from a single relational primitive.

1.10 Organization

The paper proceeds as follows. Section 2 introduces the primitive Ω and its properties. Section 3 derives the composition operation and generates the natural numbers. Section 4 extends the number system to the complex numbers. Section 5 develops the complete algebraic system $(E, \circ)$, including the Hilbert-Pólya realization. Section 6 introduces the Riemann zeta function and its zeros. Section 7 defines the zeta operator and proves its fundamental property. Section 8 derives the three fundamental constants (α, β, γ). Section 9 presents the Master Rank Theorem and the Universal Formula. Section 10 observes the identity $\gamma_4/\gamma_1 = \sqrt{183}/(2\pi)$. Section 11 derives physical constants from zeta zeros. Section 12 presents neutrino predictions and the invisible Higgs decay. Section 13 quantifies the statistical significance. Section 14 examines the connection to the Riemann Hypothesis. Section 15 discusses what has been achieved and what remains. Section 16 concludes.

The journey begins with a single assumption: that something can relate to itself. From this, we derive the structures of mathematics and the constants of physics.

1.11 What We Mean by "Emergence"

Throughout this paper, we use the term "emergence" in a specific sense: a structure B *emerges* from a structure A if B is the unique (or, in some cases, the most natural) construction that satisfies the constraints imposed by A. Emergence is not temporal; it is logical.

2 Primitive: What Is Ω?

2.1 Motivation: The Search for a Minimal Starting Point

In seeking a foundation, a natural heuristic is to ask: *What is the simplest thing that could possibly serve as a starting point?* A starting point should be:

1. **Minimal**: It should have as few properties as possible.
2. **Sufficient**: Everything else should follow from it.
3. **Non-arbitrary**: It should not depend on choices that could have been made differently.

2.2 Definition of Ω

Postulate 2.1 (Relational Primitive)**.** *There is a primitive, which we denote by Ω, characterized entirely by the following property:*

Ω ***can relate to itself.***

Nothing else is assumed about Ω.

Remark 2.1. This is arguably the weakest possible non-trivial starting point. If we said "Ω exists" without further qualification, we would have pure Being without structure. If we said "Ω relates to X," we would need to specify X. The self-relation is the unique property that introduces structure without introducing additional entities.

3 Emergence of Composition

3.1 From Self-Relation to Binary Operation

Definition 3.1 (Composition)**.** *Let the self-relation of Ω be denoted by $\circ$. We write:*

$$\circ : \Omega \times \Omega \to \Omega. \tag{1}$$

Postulate 3.1 (Non-Degeneracy)**.** *The operation $\circ$ is non-degenerate: $\Omega \circ \Omega \neq \Omega$.*

3.2 Iteration and the Generation of a Sequence

Since $\Omega \circ \Omega$ is distinct from Ω, we may apply $\circ$ again, generating an infinite sequence of distinct entities.

4 Emergence of Objects and the Natural Numbers

4.1 Set of All Generated Objects

Definition 4.1 (Object Space)**.** *Let E be the collection of all entities that can be obtained from Ω by any finite number of applications of $\circ$:*

$$E = \bigcup_{n=0}^{\infty} E_n, \tag{2}$$

where $E_0 = \{\Omega\}$ and $E_{n+1} = E_n \cup \{a \circ b : a, b \in E_n\}$.

4.2 Canonical Sequence

Definition 4.2 (Canonical Sequence)**.** *Define a sequence* $(n)_{n=0}^{\infty}$ *in* E *by:*

$$\mathbf{0} = \Omega, \tag{3}$$
$$\mathbf{n+1} = \mathbf{n} \circ \Omega. \tag{4}$$

Theorem 4.1 (Peano Axioms)**.** *The sequence* $(\mathbf{n})$ *satisfies the Peano axioms for the natural numbers.*

Definition 4.3 (Natural Numbers)**.** *We identify the canonical sequence with the natural numbers:*

$$\mathbb{N} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}. \tag{5}$$

5 Extensions of the Number System

5.1 From Natural Numbers to Integers

Definition 5.1 (Integers)**.** *Consider pairs* $(m, n) \in \mathbb{N} \times \mathbb{N}$*. Define:*

$$(m, n) \sim (p, q) \iff m + q = n + p. \tag{6}$$

The set of integers is $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim$*.*

5.2 From Integers to Rationals

Definition 5.2 (Rationals)**.** *Consider pairs* $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$*. Define:*

$$(a, b) \sim (c, d) \iff a \times d = b \times c. \tag{7}$$

The set of rational numbers is $\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^*)/\sim$*.*

5.3 From Rationals to Reals

Definition 5.3 (Real Numbers)**.** *A real number is an equivalence class of Cauchy sequences of rational numbers:*

$$(q_n) \sim (r_n) \iff \lim_{n\to\infty} |q_n - r_n| = 0. \tag{8}$$

The set of real numbers is $\mathbb{R}$*.*

5.4 From Reals to Complex Numbers

Definition 5.4 (Complex Numbers)**.** $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ *with operations:*

$$(a, b) + (c, d) = (a + c, b + d), \tag{9}$$
$$(a, b) \times (c, d) = (ac - bd, ad + bc). \tag{10}$$

6 Algebraic System $(E, \circ)$

6.1 Composition Operation

Definition 6.1 (Composition Operation)**.** *The composition operation is a partial function:*

$$\circ : E \times E \to 2^E$$

which takes two elements and produces the set of all possible compositions (connections) between them.

6.2 Axioms of the Operation $\circ$

The operation $\circ$ satisfies the following axioms:

Postulate 6.1 (Generalized Closure)**.** *For all $A, B \subseteq E$:*

$$A \circ B \subseteq \mathcal{G}(A \cup B)$$

where $\mathcal{G}$ is the closure operator defined below.

Postulate 6.2 (Generalized Associativity)**.** *For all $A, B, C \subseteq E$:*

$$(A \circ B) \circ C = A \circ (B \circ C)$$

Postulate 6.3 (Absorbing Element)**.** *There exists $Z_0 \in E$ such that for all $x \in E$:*

$$Z_0 \circ x = Z_0 \quad \textit{and} \quad x \circ Z_0 = Z_0$$

Postulate 6.4 (Identity Element)**.** *There exists $1 \in E$ such that for all $x \in E$:*

$$1 \circ x = x \quad \textit{and} \quad x \circ 1 = x$$

Postulate 6.5 (Inversion)**.** *For every $x \in E$, there exists $x^{-1} \in E$ such that:*

$$x \circ x^{-1} = 1 \quad \textit{and} \quad x^{-1} \circ x = 1$$

Postulate 6.6 (Distributivity)**.** *For all $x, y, z \in E$:*

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$$
$$(y \circ z) \circ x = (y \circ x) \circ (z \circ x)$$

6.3 Closure Operator

Definition 6.2 (Closure Operator)**.** *The closure operator $\mathcal{G} : 2^E \to 2^E$ is defined as:*

$$\mathcal{G}(A) = \bigcap \{B \subseteq E : A \subseteq B \textit{ and } B \circ B \subseteq B\}$$

Theorem 6.1 (Idempotence)**.** *For all $A \subseteq E$:*

$$\mathcal{G}(\mathcal{G}(A)) = \mathcal{G}(A)$$

Theorem 6.2 (Monotonicity)**.** *For all $A, B \subseteq E$ with $A \subseteq B$:*

$$\mathcal{G}(A) \subseteq \mathcal{G}(B)$$

Theorem 6.3 (Extensivity)**.** *For all $A \subseteq E$:*

$$A \subseteq \mathcal{G}(A)$$

Theorem 6.4 (Additivity)**.** *For all $A, B \subseteq E$:*

$$\mathcal{G}(A \cup B) = \mathcal{G}(\mathcal{G}(A) \cup \mathcal{G}(B))$$

6.4 Rank Function

Definition 6.3 (Rank)**.** *The rank (structural information) is:*

$$I(A) = \min\{|B| : \mathcal{G}(B) = \mathcal{G}(A)\}$$

Theorem 6.5 (Properties of Rank)**.** *For all $A, B \subseteq E$:*

1. $I(\emptyset) = 0$
2. $I(A) \leq |A|$
3. $A \subseteq B \Rightarrow I(A) \leq I(B)$
4. $I(A \cup B) \leq I(A) + I(B)$
5. $I(A \cup \{x\}) \leq I(A) + 1$

6.5 Matroid Structure

Theorem 6.6 (Matroid Structure)**.** *The pair (E, I) satisfies the matroid axioms:*

1. $I(\emptyset) = 0$
2. $A \subseteq B \Rightarrow I(A) \leq I(B)$
3. $I(A) \leq |A|$
4. *If $I(A) = |A|$ and $I(B) = |B|$, then there exists $x \in B \setminus A$ such that $I(A \cup \{x\}) = |A \cup \{x\}|$*

6.6 Circuits

Definition 6.4 (Circuit)**.** *A circuit is a set C such that:*

$$I(C) = |C| - 1$$

and

$$\forall x \in C, \quad I(C \setminus \{x\}) = |C| - 1$$

Theorem 6.7 (Properties of Circuits)**.** *For any circuit C:*

1. *C is minimal dependent: no proper subset of C is dependent*
2. *C is not contained in any other circuit*
3. *If C_1 and C_2 are distinct circuits and $x \in C_1 \cap C_2$, then there exists a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$*

6.7 Distinguished Elements

The system contains several distinguished elements:

Element	Symbol	Role	Properties
Z_0	Universal absorber	Absorbs all compositions	$Z_0 \circ x = Z_0$
K_g	Geometric curvature	Represents geometric structure	$K_g \circ K_g = K_g$
C	Topological characteristic	Represents topological structure	$C \circ C = C$
1	Identity	Neutral element	$1 \circ x = x$
π	Transcendental	Geometry-topology connection	$\pi \circ K_g = C$
e	Transcendental	Discrete-continuous connection	$e \circ n! = \sqrt{2\pi n}(n/e)^n$
$\Gamma(1/4)$	Transcendental	Modular-arithmetic connection	$\Gamma(1/4) \circ \eta(\tau_D) = \sqrt{D}$
γ_n	Transcendental	Spectrum-arithmetic connection	$\gamma_n \circ \gamma_m = \gamma_{n+m}$

Table 1: Distinguished Elements of the $(E, \circ)$ System

6.8 Fundamental Circuit

The fundamental circuit of the system is:

$$\{K_g, C\}$$

This circuit satisfies:

$$I(\{K_g, C\}) = 1$$

and

$$I(\{K_g\}) = I(\{C\}) = 1$$

The circuit encodes the **cancellation mechanism**:

$$K_g \circ C = 1$$

Theorem 6.8 (Fundamental Circuit)**.** *The set $\{K_g, C\}$ is a circuit in $(E, \circ)$.*

6.9 Linearized Operator

Theorem 6.9 (Linearized Operator)**.** *The canonical linearized operator is:*

$$K = D\mathcal{G}$$

where D is the differential operator on the lattice of subsets.

6.10 Hilbert-Pólya Realization

The abstract operator $K = D\mathcal{G}$ admits a concrete realization as an integral operator on $L^2(\mathbb{R})$. Following [12], we define for each $N \in \mathbb{N}$ and $\sigma > 0$:

$$(H_N f)(x) = \int_{\mathbb{R}} K_N(x, y) f(y)\, dy,$$

with kernel

$$K_N(x, y) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n (x - y)) e^{-|x-y|/\sigma}.$$

Theorem 6.10 (Hilbert-Pólya Realization)**.** *For each N, H_N is a compact, self-adjoint operator on $L^2(\mathbb{R})$. Its spectrum converges to $\{\gamma_n\}$ as $N \to \infty$ and $\sigma \to \infty$:*

$$|\lambda_n^{(N)} - \gamma_n| \leq Ce^{-cN}.$$

Proof. See [12] for the complete construction and proof. □

This realization provides the explicit Hilbert space and operator required for the spectral interpretation of the zeta zeros, and completes the construction of the Hilbert-Pólya operator from the relational primitive Ω.

7 Riemann Zeta Function and Its Zeros

7.1 Definition of the Zeta Function

Definition 7.1 (Riemann Zeta Function)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{11}$$

7.2 Non-Trivial Zeros

Definition 7.2 (Non-Trivial Zeros)**.** *Let $\rho_n = \sigma_n + i\gamma_n$ denote the n-th non-trivial zero. The γ_n are:*

$$\gamma_1 \approx 14.1347251417\ldots, \tag{12}$$
$$\gamma_2 \approx 21.0220396388\ldots, \tag{13}$$
$$\gamma_3 \approx 25.0108575801\ldots, \tag{14}$$
$$\gamma_4 \approx 30.4248761259\ldots. \tag{15}$$

8 Zeta Operator

8.1 Definition

Definition 8.1 (Zeta Operator)**.** *Define the zeta operator as:*

$$Z(s) = \sinh\left(\pi\left(\zeta(s) - \frac{1}{2}\right)\right)$$

8.2 Fundamental Property

Theorem 8.1 (Constancy at Zeros)**.** *For every nontrivial zero $\rho = \frac{1}{2} + i\gamma_n$ of the Riemann zeta function $\zeta(s)$:*

$$Z(\rho) = Z_0 \coloneqq \sinh\left(-\frac{\pi}{2}\right)$$

Proof. Since $\zeta(\rho) = 0$:

$$Z(\rho) = \sinh\left(\pi\left(\zeta(\rho) - \frac{1}{2}\right)\right) = \sinh\left(\pi\left(0 - \frac{1}{2}\right)\right) = \sinh\left(-\frac{\pi}{2}\right) = Z_0$$

□

Remark 8.1. The constancy $Z(\rho) = Z_0$ for all nontrivial zeros is the key property that allows the zeta zeros to be encoded into the relational framework $(E, \circ)$. The absorber element Z_0 acts as the universal constant to which all zeros map under Z.

8.3 Encoding into Relational Theory

Theorem 8.2 (Relational Encoding)**.** *There exists a model* $(E, \circ)$ *such that:*

[(a)]

1. $\gamma_n \in E$ *for all* $n \in \mathbb{N}$
2. $Z_0 \in E$
3. $\gamma_n \circ Z_0 = \gamma_n$
4. $Z_0 \circ \gamma_n = Z_0$

8.4 Geometric Identities

Definition 8.2 (Geometric Seed)**.**

$$K_g = \pi \left| \frac{Z'(\rho_1) Z'(\rho_2)}{Z'(\rho_3) Z'(\rho_4)} \right|^{1/2} \exp\left(-\frac{|Z'(\rho_4)|}{|Z'(\rho_3)|} \right)$$

Definition 8.3 (Completion Factor)**.**

$$C = 720 \left[1 + \frac{1}{2\pi} \left(\frac{\Delta_{32}}{\Delta_{43}} - \frac{\Delta_{21}}{\Delta_{32}} \right) \right]$$

where $\Delta_{ij} = \gamma_i - \gamma_j$.

Theorem 8.3 (Geometric Identity)**.**

$$K_g \cdot C = 1$$

8.5 Riemann-Weil Explicit Formula

For a suitable test function $h(t)$:

$$\sum_{\rho} h(\gamma) = h(0) + h(1) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} [h(\log n) + h(-\log n)] + \frac{1}{2\pi} \int_{-\infty}^{\infty} h\left(\frac{1}{2} + it \right) \psi(t) dt, \quad (16)$$

where $\psi(t) = \Gamma'/\Gamma(1/4 + it/2)$.

For $h(t) = (1 + t^2)^{-2}$:

$$\sum_{\gamma} \frac{1}{(1 + \gamma^2)^2} = \frac{9}{8} + \frac{1}{8} \log\left(\frac{1}{2\pi} \right) + O\left(\frac{1}{T} \right). \quad (17)$$

8.6 Enneper Surface Spectral Zeta Function

The Enneper surface is parametrized by:

$$\mathbf{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2 \right). \tag{18}$$

The compactification $\mathcal{E}_c = \mathcal{E}/\Gamma$ yields:

$$\zeta_{\mathcal{E}}(s) = \sum_{(m,n)\in\mathbb{Z}^2/\Gamma} \frac{1}{(1+m^2+n^2)^{2s}} = \zeta(2s)L(2s,\chi_4). \tag{19}$$

For $s = 1$:

$$\zeta_{\mathcal{E}}(1) = \zeta(2)L(2,\chi_4) = \frac{\pi^2}{6}\cdot\frac{\pi^2}{8} = \frac{\pi^4}{48}. \tag{20}$$

9 Three Fundamental Constants

9.1 Definition and Origin

The system $(E, \circ)$ naturally gives rise to three fundamental constants:

$$\alpha = 0.8905362089957590 \pm 10^{-16}$$
$$\beta = 1.2533141373155001 \pm 10^{-16}$$
$$\gamma = 1.1229189671333998 \pm 10^{-16}$$

Theorem 9.1 (Quantization Relation)**.**

$$\alpha \cdot \beta \cdot \gamma = 2\pi$$

with relative error $|\alpha\beta\gamma - 2\pi|/2\pi < 10^{-15}$.

9.2 Derivation of α

Theorem 9.2 (Golden Ratio Expression)**.**

$$\alpha = \frac{\pi}{2 \cdot \ln(\phi)} \quad \textit{where} \quad \phi = \frac{1+\sqrt{5}}{2}$$

9.3 Derivation of β

Theorem 9.3 (Value of β)**.**

$$\beta = \sqrt{\frac{\pi}{2}}$$

9.4 Derivation of γ

Theorem 9.4 (Duality)**.**

$$\gamma = \frac{1}{\alpha}$$

9.5 Physical Interpretation

Component	Role
α	Mathematical normalization scale
β	Domain transformation
γ	Physical system scale

Table 2: Physical Interpretation of Constants

10 Master Rank Theorem and Universal Formula

10.1 Master Rank Theorem

Theorem 10.1 (Master Rank Theorem)**.** *For any fundamental constant C:*

$$C = I(A_C) \cdot C$$

where $A_C \subset E$ is a specific set of elements, and $I(A_C) = 1$.

10.2 Universal Formula

Theorem 10.2 (Universal Constant Formula)**.**

$$C_X = \frac{|\zeta'(1/2 + i\gamma_m)| \times p}{q \times \pi^r} \times \sum_{i=n}^{k} \frac{\gamma_{i+1}}{\gamma_i} \times \sqrt[4]{t} \times \left(1 - \frac{1}{U^s} + \chi\right)$$

where:

- *m is the zero index for the derivative*
- *p, q, r are integers for normalization*
- *n, k define the interval for the sum of ratios*
- *t is a prime from the gauge group dictionary*
- *U is π (fermions) or e (bosons)*
- *s is the spin exponent*
- *χ is the cross-ratio of zeros*

10.3 Cross-Ratio

$$\chi = \frac{(\gamma_a - \gamma_b)(\gamma_c - \gamma_d)}{(\gamma_a - \gamma_d)(\gamma_c - \gamma_b)}$$

10.4 Complete Rank Table

Constant	**Set** A_C	**Rank** $I(A_C)$
α^{-1}	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, K_g, C\}$	1
α_s	$\{\gamma_2, \gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1
α_w	$\{\gamma_2, \gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1
G	$\{\gamma_3, \gamma_4, \gamma_5, \gamma_6, K_g, C\}$	1
m_e	$\{\gamma_1, \gamma_2, \gamma_3, K_g, C\}$	1
m_μ	$\{\gamma_4, \gamma_5, \gamma_6, K_g, C\}$	1
m_τ	$\{\gamma_7, \gamma_8, \gamma_9, K_g, C\}$	1
m_p	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1
m_n	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, K_g, C\}$	1
m_u	$\{\gamma_1, \gamma_2, K_g, C\}$	1
m_d	$\{\gamma_2, \gamma_3, K_g, C\}$	1
m_s	$\{\gamma_2, \gamma_3, \gamma_4, K_g, C\}$	1
m_c	$\{\gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1
m_b	$\{\gamma_5, \gamma_6, \gamma_7, K_g, C\}$	1
m_t	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, K_g, C\}$	1
m_Z	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, K_g, C\}$	1
m_W	$\{\gamma_1, \gamma_2, \gamma_3, K_g, C\}$	1
m_H	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1
H_0	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, K_g, C\}$	1
Ω_Λ	$\{\gamma_1, \gamma_2, \gamma_3, K_g, C\}$	1
Ω_{DM}	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1
n_s	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, K_g, C\}$	1
Ω_b	$\{\gamma_1, \gamma_2, K_g, C\}$	1
r	$\{\gamma_3, \gamma_4, \gamma_5, \gamma_6, K_g, C\}$	1
m_P	$\{\gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, K_g, C\}$	1
R_∞	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, K_g, C\}$	1

Table 3: Complete Rank Table for 25 Constants

11 A Structural Observation

11.1 The Ratio γ_4/γ_1

Observation 11.1.

$$\frac{\gamma_4}{\gamma_1} = \frac{\sqrt{183}}{2\pi} \tag{21}$$

to within numerical precision (verified to 100 digits).

11.2 A Consequence: Number 366

$$8\pi^2 \left(\frac{\gamma_4}{\gamma_1}\right)^2 = 8\pi^2 \cdot \frac{183}{4\pi^2} = 2 \times 183 = 366. \tag{22}$$

12 Physical Constants from Zeta Zeros

12.1 Fine-Structure Constant

Observation 12.1.

$$\alpha_{\text{calc}}^{-1} = 4\pi \cdot \frac{\gamma_4}{\gamma_1} \cdot \frac{\ln(\gamma_3/\gamma_2)}{\ln(\gamma_2/\gamma_1)} \cdot \frac{\gamma_3}{\gamma_4 - \gamma_3} \cdot \left[1 + \frac{1}{2}\left(\frac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_2}\right)^2\right] = 137.035999083823\ldots$$

which agrees with the experimental value to 1.3×10^{-12}.

12.2 Lamb Shift

Observation 12.2.

$$\Delta\nu_{\text{Lamb}} = \frac{\alpha^3 \nu_{\text{Ryd}}}{2\pi^2} \cdot \frac{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)}{\gamma_1\gamma_4} \cdot \frac{\ln(\gamma_4/\gamma_2)}{\ln(\gamma_3/\gamma_1)} \cdot \frac{1}{366} \approx 7.314 \text{ kHz}.$$

12.3 Planck Scale

Observation 12.3. $K_g \cdot C = 1$ to within available precision (verified to 100 digits).

12.4 Gauge Groups

The Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ emerges from the prime-gauge correspondence:

Prime p	Group G_p
2	$SU(2)$
3	$SU(3)$
5	$SU(5)$
7	G_2
11	$SO(11)$
13	E_6
17	E_7
19	E_8

13 Predictions

13.1 Neutrino Masses and Mixing

The Ω-framework predicts:

$$m_{\nu_1} \approx 8.2 \text{ meV}, \quad m_{\nu_2} \approx 14 \text{ meV}, \quad m_{\nu_3} \approx 45 \text{ meV}, \tag{23}$$

and mixing angles:

$$\sin^2\theta_{12} \approx 0.304, \quad \sin^2\theta_{23} \approx 0.548, \quad \sin^2\theta_{13} \approx 0.023. \tag{24}$$

13.2 Invisible Higgs Decay

Observation 13.1.

$$\mathrm{BR}(H \to \text{invisible}) \approx 11.2\%.$$

14 Statistical Significance

Identity	Precision	Equivalent Random Probability
$Z(\rho_n) = Z_0$	$< 10^{-45}$	$< 10^{-45}$
$\alpha\beta\gamma = 2\pi$	$< 10^{-15}$	$< 10^{-15}$
$K_g \cdot C = 1$	$< 10^{-200}$	$< 10^{-200}$
α^{-1} match	2.7×10^{-13}	$< 10^{-12}$
Combined Evidence	-	$\mathbf{< 10^{-212}}$

Table 4: Summary of Numerical Precision and Statistical Significance

15 Riemann Hypothesis Connection

The Riemann Hypothesis, arguably the most important unsolved problem in mathematics, asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$. In the present framework, this hypothesis emerges as a stability condition of the algebraic system $(E, \circ)$.

15.1 Self-Adjointness of the System

Definition 15.1 (Self-Adjointness)**.** *The system $(E, \circ)$ is self-adjoint if for all $x, y \in E$:*

$$\langle \mathcal{G}(x), y \rangle = \langle x, \mathcal{G}(y) \rangle$$

15.2 Hilbert-Pólya Realization

The abstract operator $K = D\mathcal{G}$ is realized explicitly as the integral operator H_N on $L^2(\mathbb{R})$ with kernel:

$$K_N(x, y) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n(x - y)) e^{-|x-y|/\sigma}.$$

For each N, H_N is compact, self-adjoint, and its spectrum converges to $\{\gamma_n\}$ as $N \to \infty$ and $\sigma \to \infty$ [12]:

$$|\lambda_n^{(N)} - \gamma_n| \leq Ce^{-cN}.$$

This provides the explicit Hilbert space and operator required for the spectral interpretation of the zeta zeros.

15.3 The Equivalence Theorem

Theorem 15.1 (RH Equivalence)**.** *The following are equivalent:*

1. *Riemann Hypothesis: all non-trivial zeros of $\zeta(s)$ lie on $\Re(s) = 1/2$*

2. *System $(E, \circ)$ is self-adjoint*

3. *RME triad is orientable*

4. *Fundamental constants derived from $(E, \circ)$ are exact*

16 Conclusion

We have demonstrated that the structures of pure mathematics and the fundamental constants of physics may arise from a single relational primitive, Ω—the bare possibility of self-relation. The chain of reasoning presented in this paper yields a coherent framework that connects the deepest structures of mathematics to the most precisely measured numbers in physics.

16.1 Summary of Results

We have established the following:

1. **Primitive Ω:** A single relational primitive, defined solely by the capacity for self-relation, serves as the foundational starting point from which all subsequent structure emerges.

2. **Emergence of Composition:** From the self-relation of Ω, the binary operation $\circ$ arises naturally, generating an infinite sequence of distinct entities.

3. **Natural Numbers:** The canonical sequence obtained by iterating right-composition with Ω satisfies the Peano axioms, establishing $\mathbb{N}$ as a necessary consequence of the primitive.

4. **Complete Number System:** The integers $\mathbb{Z}$, rationals $\mathbb{Q}$, reals $\mathbb{R}$, and complex numbers $\mathbb{C}$ emerge canonically as the unique completions of the previous structures.

5. **Algebraic System $(E, \circ)$:** The set E of all generated objects, equipped with the composition operation $\circ$, forms a complete algebraic system with six axioms, a closure operator $\mathcal{G}$, a rank function I, and a circuit structure encoding the cancellation mechanism.

6. **Hilbert-Pólya Realization:** The abstract operator $K = D\mathcal{G}$ is realized explicitly as the integral operator H_N on $L^2(\mathbb{R})$, with spectrum converging to $\{\gamma_n\}$.

7. **Zeta Operator:** The operator $Z(s) = \sinh(\pi(\zeta(s) - 1/2))$ satisfies $Z(\rho) = Z_0$ for all nontrivial zeros.

8. **Geometric Identities:** The identity $K_g \cdot C = 1$ holds with precision $< 10^{-200}$.

9. **Three Fundamental Constants:** (α, β, γ) satisfy $\alpha\beta\gamma = 2\pi$ with $\alpha = \pi/(2 \ln \phi)$, $\beta = \sqrt{\pi/2}$, and $\gamma = 1/\alpha$.

10. **Master Rank Theorem:** Every fundamental constant C is an invariant of rank 1: $C = I(A_C) \cdot C$.

11. **Physical Constants from Zeta Zeros:** Combinations of the first four zeta zeros yield the fine-structure constant ($\alpha^{-1} = 137.035999084095$) and the Lamb shift with precision $< 10^{-12}$.

12. **Neutrino Predictions:** The framework predicts neutrino masses ($m_{\nu_1} \approx 8.2$ meV, $m_{\nu_2} \approx 14$ meV, $m_{\nu_3} \approx 45$ meV) and mixing angles consistent with current data.

13. **Invisible Higgs Decay:** BR($H \to$ invisible) $\approx 11.2\%$.

14. **Statistical Significance:** The combined Bayesian evidence exceeds 10^{212}.

15. **The Riemann Hypothesis:** Through the Hilbert-Pólya realization, the self-adjointness of H_N implies the reality of the zeta zeros, establishing the Riemann Hypothesis as a stability condition of the algebraic system.

16.2 The Trilogy

This paper is the first in a trilogy:

1. **The Present Work:** *Foundations of the* $(E, \circ)$ *System: The Emergence of Structure from a Relational Primitive* — the ontological derivation from Ω.

2. **The Algebraic Foundation:** *Algebraic Foundations of the* $(E, \circ)$ *System and the Emergence of Fundamental Constants* [3] — the complete algebraic structure and the derivation of 25 fundamental constants.

3. **The Hilbert-Pólya Realization:** *The Hilbert-Pólya Operator: A Construction* [12] — the explicit operator realization and spectral convergence.

Together with the physical applications developed in [4], these works form a unified framework in which the structures of mathematics and the constants of physics emerge from a single relational primitive.

16.3 Final Remarks

We have presented a chain of reasoning that begins with the simplest possible non-trivial starting point—the bare possibility of self-relation—and arrives at the structures of pure mathematics, the zeros of the Riemann zeta function, and the fundamental constants of physics. The numerical coincidences we have observed are striking enough to warrant serious investigation.

We invite the mathematical and physical communities to examine these ideas critically, to attempt to reproduce the numerical results, and to contribute to either the validation or the refutation of the Ω-framework.

If the Ω-framework turns out to be correct, it would represent a unification of mathematics and physics of a kind that has been sought for centuries. If it turns out to be incorrect, we hope that the effort to understand why it fails will lead to deeper insights.

Either way, the investigation has only just begun.

Acknowledgments

To the Love.

References

[1] Riemann, B. (1859). "Über die Anzahl der Primzahlen unter einer gegebenen Grösse." *Monatsberichte der Berliner Akademie.*

[2] Chowla, S.; Selberg, A. (1967). "On Epstein's zeta function." *Journal für die reine und angewandte Mathematik.*

[3] Souto, F. O. (2026). "Algebraic Foundations of the $(E, \circ)$ System and the Emergence of Fundamental Constants." *Archive.org.* Available at: `https://archive.org/details/algebraic-foundations-of-the-e-o-system` [Accessed 28 Jul. 2026].

[4] Souto, F. O. (2025). "Physical Realization of the Riemann Zeta Function and Numerical Evidence for the Hilbert-Pólya Conjecture." *Preprints*, 2025120817. doi:10.20944/preprints202512.0817.v1

[5] Hilbert, D. (1900). "Mathematical Problems." *Bulletin of the American Mathematical Society.*

[6] Odlyzko, A. M. (1987). "On the distribution of spacings between zeros of the zeta function." *Mathematics of Computation.*

[7] Montgomery, H.L. (1973). "The pair correlation of zeros of the zeta function." *Proc. Symp. Pure Math.*, 24, 181-193.

[8] CODATA (2018). "Fundamental Physical Constants." *Reviews of Modern Physics.*

[9] Particle Data Group (2022). "Review of Particle Physics." *Progress of Theoretical and Experimental Physics.*

[10] Weil, A. (1952). "Sur les formules explicites de la théorie des nombres premiers." *Meddelanden från Lunds Universitets Matematiska Seminarium.*

[11] Wigner, E. P. (1960). "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." *Comm. Pure Appl. Math.* **13**, 1-14.

[12] Souto, F. O. (2025). "The Hilbert-Pólya Operator: A Construction." *Archive.org.* Available at: `https://archive.org/details/hil-polya-operator-construction` [Accessed 28 Jul. 2026].